Exam 2, November 3, 2015.
Math 308

Instructions: All questions are worth the same number of points. **Important: No books, calculators, or notes are allowed. Turn off cell phones, alarms, and anything else that makes noises!** You must show **all** your work to receive credit. Any crossed out work will be disregarded (even if correct). Write **one** clear answer with a coherent derivation for each question. Good luck!

Name: Solutions

[**1**] Prove or disprove: for all $n \in \mathbb{N}$, $3|(4^n - 1)$.

Proof: We proceed by induction.

- The base case, when $n = 1$, is clear, since $3|4 - 1$.
- Assume that for $k \in \mathbb{N}$, $3|(4^k - 1)$. Then, we write $4^{k+1} - 1 = (3 + 1)4^k - 1 = 4^k - 1 + 3(4^k)$. Since 4^k is an integer, we see that $3|4^{k+1} - 1$.

[**2**] Prove or disprove: there exist positive integers m and n such that

$$m^2 - n^2 = 1.$$

We will prove the following:

Proposition: For all positive integers m and n, $m^2 - n^2 \neq 1$.

Proof: Suppose that there exist positive integers m and n such that $m^2 - n^2 = 1$. So, 1 can be written as $(m + n)(m - n)$. Since both m and n are positive, the only possibility is that $m + n = 1$ and $m - n = 1$: those two equations imply that $n = 0$, which is a contradiction.

[**3**] The sequence $\{a_n\}$ is defined recursively by $a_1 = 2$, $a_2 = 8$, and

$$a_n = 4a_{n-1} - 3a_{n-2},$$

for $n \geq 3$. Prove that $a_n = 3^n - 1$ for all $n \in \mathbb{N}$.

Proof: We proceed by induction.

- The base case, when $n = 3$, is clear: $a_3 = 4a_2 - 3a_1 = 26 = 3^3 - 1$.

- Let $k \in \mathbb{N}$, $k > 3$, and assume that for all $j \in \mathbb{N}$, $3 \le j \le k$, $a_j = 3^j - 1$.

  Then,

$$\begin{aligned} a_{k+1} &= 4a_k - 3a_{k-1} \\ &= 4(3^k - 1) - 3(3^{k-1} - 1) \\ &= (4-1)3^k - (4-3) = 3^{k+1} - 1. \end{aligned}$$

By the Strong Principle of Induction, we see that $a_n = 3^n - 1$ for all $n \in \mathbb{N}$.

[4] Prove that $\sqrt{5}$ is an irrational number.

Proof: By contradiction, assume $\sqrt{5} = p/q$, for two nonzero integers p and q. We may assume that p and q have no common factors.

Hence, $p^2 = 5q^2$, and so $5|p^2$. This implies that $5|p$, that is, there exists an integer k such that $p = 5k$. This last equation implies that $25k^2 = 5q^2$, which tells us that $5|q^2$, and ultimately, that $5|q$. This contradicts the assumption that p and q are coprimes.

[5]

(a) What is the definition of a well-ordered set?

A set $A \subset \mathbb{R}$ is *well-ordered* if each nonempty subset $S \subset A$ contains a least element.

(b) Consider the set $A = \{x \in \mathbb{Q}; x \ge 0\}$. Is this set well-ordered? Prove your answer.

This set is not well-ordered: it contains the subset $S = \{x \in \mathbb{Q}; x > 0\}$, which is nonempty and does not admit a least element. We see this by assuming that S has a least element, say ℓ. Since $\ell \in S$, ℓ is a positive rational number. Hence, the number $\ell/2$ is also rational and positive, which implies that $\ell/2 \in S$ as well. This contradicts the fact that ℓ was the least element of the set S.